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Investigation of elliptical waveguides integrated with negative refractive index materials

A.K. Shahi

Department of Physics, Ram Lalit Singh P.G. College, Kailhat,Narayanpur - 231305, India

Abstract: This paper investigates the modal characteristics of an elliptical core optical fiber embedded with negative refractive index material (NRIM). Utilizing the method of separation of variables in the elliptical coordinate system, we derive the eigenvalue equation in terms of first and second-kind Mathieu functions under the weak guidance condition. Numerical computations focus on cutoff frequencies and the behaviour of lower-order odd TM modes for different aspect ratios. The study reveals that the proposed waveguide supports only odd TM modes when highlighting a unique mode-selective feature. Additionally, the cutoff frequency exhibits a decreasing trend as the aspect ratio increases, demonstrating a direct correlation between waveguide geometry and modal dispersion. The variation of cutoff V-values is systematically analyzed and graphically represented, providing insights into optimizing waveguide design. These findings contribute to the advancement of optical waveguide technologies, offering potential applications in high- precision sensing, optical communication, and photonic integrated circuits.

1. INTRODUCTION

Elliptical waveguides have been widely studied due to their unique modal properties and potential applications in optical communication and sensing technologies. Unlike circular waveguides, elliptical waveguides exhibit anisotropic confinement, leading to distinct dispersion characteristics and polarization-dependent behavior. Theoretical and experimental research on elliptical waveguides has provided valuable insights into their modal propagation and cutoff conditions [1-3].

The integration of negative refractive index materials (NRIMs) in optical waveguides has recently gained significant interest due to their ability to manipulate electromagnetic waves in ways not possible with conventional materials. NRIMs exhibit negative permittivity and permeability, leading to phenomena such as negative refraction, backward wave propagation, and enhanced mode confinement [4-7]. Early studies on guided optical waves in fibers with negative dielectric constants have demonstrated their potential in modifying modal dispersion and cutoff characteristics [8]. Additionally, metamaterials with engineered NRIM properties have been successfully incorporated into waveguide structures, enabling novel waveguiding behaviors [9-10].

Several studies have explored the modal properties of elliptical waveguides, particularly in the presence of metamaterials and anisotropic media. For instance, research on elliptical dielectric waveguides [2] and highly birefringent elliptical-core fibers [11] has highlighted the significance of structural geometry in controlling wave propagation. Furthermore, analytical and numerical investigations have provided insights into cutoff wave numbers and polarization-dependent modal dispersion in elliptical waveguides loaded with non-traditional materials [12-13]. However, the impact of negative refractive index material loading on elliptical core optical fibers remains an area that requires further exploration.

In this paper, we investigate the modal characteristics of an elliptical core optical fiber embedded with NRIM. Using the method of separation of variables in an elliptical coordinate system, we derive the eigenvalue equation in terms of Mathieu functions under the weak guidance condition. Numerical computations focus on cutoff frequencies and the behavior of lower-order odd TM modes for different aspect ratios. Our results reveal that the proposed waveguide selectively supports odd TM modes, exhibiting a mode-selective feature that distinguishes it from conventional elliptical waveguides. Additionally, we observe a decreasing trend in cutoff frequency as the aspect ratio increases, indicating a strong correlation between waveguide geometry and modal dispersion. The variation of cutoff V-values is systematically analyzed and graphically represented to provide insights into optimizing waveguide design.

Historically, studies on elliptical waveguides have played a crucial role in shaping waveguide technology. Early works, such

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as those by L.-J. Chu (1938) [14], laid the foundation for understanding wave propagation in elliptic hollow waveguides, demonstrating their distinct modal properties compared to circular waveguides. Later, Shaw et al. (1991) [15] conducted **a** numerical analysis of dual-mode elliptical-core fibers, showing the feasibility of mode control in elliptical structures. Additionally, research by Kim et al. (1987) [16] explored the use of highly elliptical-core fibers for two-mode fiber devices, highlighting their relevance in optical communication. These studies reinforce the significance of elliptical waveguides in photonic applications and provide a strong theoretical basis for our findings. The findings presented in this study contribute to the advancement of optical waveguide technologies, with potential applications in high-precision optical sensing, optical communication, and photonic integrated circuits. The integration of negative refractive index materials into elliptical waveguides opens up new possibilities for engineered modal confinement and tunable waveguide properties, which can be leveraged for next-generation optical devices [17-21].

The integration of NRIMs into elliptical waveguides opens up new possibilities for enhanced modal confinement, tunable dispersion, and engineered cutoff conditions, which are crucial for applications in optical sensing, photonic integrated circuits, and high-precision communication systems. Future research could explore higher-order mode interactions, temperature-dependent refractive index variations, and additional NRIM compositions to further optimize the waveguide design for practical applications

2. MATHEMATICAL MODELING

The elliptical waveguide presents a departure from circular symmetry, requiring a distinct approach to analyzing wave propagation. To understand its characteristics, we first examine the elliptical coordinate system, an orthogonal framework similar to the circular and Cartesian coordinate systems. The elliptical coordinates (ξ,η) which are related to Cartesian coordinate system as

$$x = q\cosh\xi\cos\eta \qquad (1a)$$

$$y = q\sinh\xi\sin\eta \qquad (1b)$$

where q is semi-focal distance of the family of confocal ellipse defined by ξ=constant. Let us consider one such ellipse let $\xi = \xi_0$, having core refractive index $-n_1$ and clad refractive index n_2 as shown in the Fig.1 and Fig.2. Let the major and minor axes of the ellipse are $2a$ and $2b$ respectively, then a and b are given by

$a = q$

b $=q$

The eccentricity of this ellipse is given by

$$e^2 = 1 - \left(\frac{b}{a}\right)^2 = \text{sech}^2\xi_0$$

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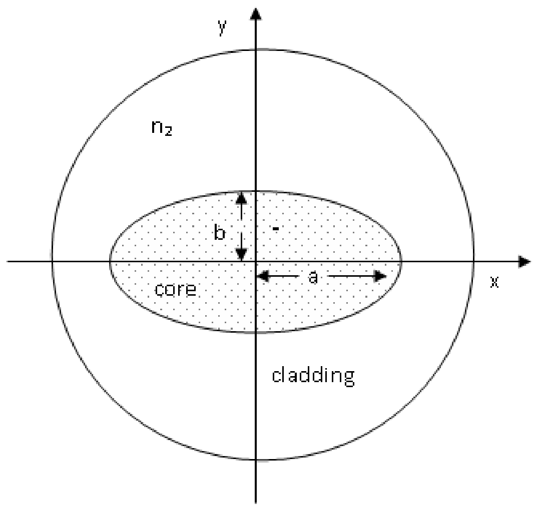

Fig. 1: Cross-sectional view of negative refractive index material loaded elliptical core fiber.

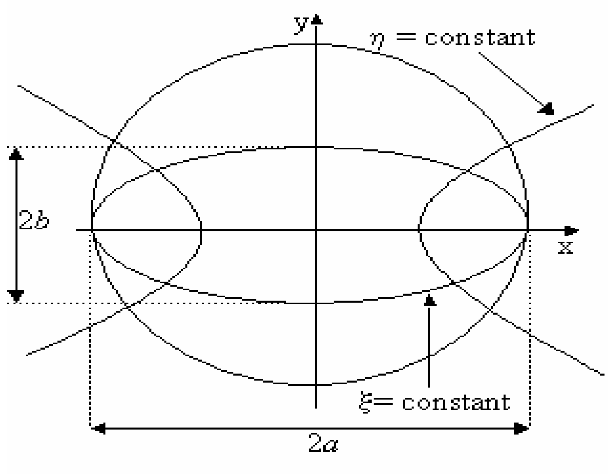

Fig. 2: The elliptical-coordinate system.

Now let us see wave equation in elliptical coordinate system: wave equation can be written as:

$\nabla^2\phi = \frac{1}{v^2}\frac{\partial^2\phi}{\partial t^2}$, and we can write $\nabla^2\phi$ in any orthogonal coordinate system as

$$\nabla^2\phi = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2h_3}{h_1}\frac{\partial\phi}{\partial u_1}\right)+\frac{\partial}{\partial u_2}\left(\frac{h_3h_1}{h_2}\frac{\partial\phi}{\partial u_2}\right)+\frac{\partial}{\partial u_3}\left(\frac{h_1h_2}{h_3}\frac{\partial\phi}{\partial u_3}\right)\right] \quad (2)$$

where h_1, h_2, h_3 are known as weighting factor. For elliptical coordinates:

$$\frac{1}{h_\xi}=\frac{1}{h_1}=q(\sinh^2\xi+\sin^2\eta)^{\frac{1}{2}}=q(\cosh^2\xi-\cos^2\eta)^{\frac{1}{2}}=q\left(\frac{1}{2}(\cosh 2\xi-\cos 2\eta)\right)^{\frac{1}{2}} \quad (3a)$$

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$$h_\eta = h_2 = q(\sinh^2\xi+\sin^2\eta)^{\frac{1}{2}} = q(\cosh^2\xi-\cos^2\eta)^{\frac{1}{2}} = q\left(\frac{1}{2}(\cosh 2\xi-\cos 2\eta)\right)^{\frac{1}{2}} \quad (3b)$$

$$h_\rho = 1 = h_3 \quad (3c)$$

$u_1=\xi,\quad u_2=\eta,\quad u_3=\rho$ and $\frac{1}{v^2}=(\varepsilon\mu)=n^2$

Now after getting basic idea of elliptical coordinate system let us see the propagation of electromagnetic wave in this medium when filled with a material having negative refractive index.

2.1 Propagation of electromagnetic wave in elliptical fiber

The propagation of electromagnetic wave in a medium is given by solving Maxwell equation for that medium. Here we are starting with the wave equation:

$$\nabla^2\phi = \frac{1}{v^2}\frac{\partial^2\phi}{\partial t^2}$$

Now expressing this wave equation in elliptical coordinate as given in equation (2) and putting values of $h1, h2, h3$ from equation (3a), (3b), (3c) and solving using simple mathematics we get *:*

$$\nabla^2\phi = \frac{1}{(\sinh^2\xi+\sin^2\eta)}\left[\frac{\partial^2\phi}{\partial\xi^2}+\frac{\partial^2\phi}{\partial\eta^2}+\frac{q^2}{2}(\cosh 2\xi-\cos 2\eta)\frac{\partial^2\phi}{\partial\rho^2}\right] = -n_j^2k^2 \quad 4(a)$$

After rearranging the above equation, we get:

$$\left[\frac{\partial^2\phi}{\partial\xi^2}+\frac{\partial^2\phi}{\partial\eta^2}+\frac{q^2}{2}(n_j^2k^2-\beta^2)(\cosh 2\xi-\cos 2\eta)\right]=0 \quad (4b)$$

where, j = 1 and 2, j = 1 is for the core region and j = 2 is for the cladding region.

Now we use variable separable method to solve above equation. Let us assume $\phi(\xi,\eta)=f(\xi)g(\eta)$ and put in equation (4b) and dividing both side by $f(\xi)g(\eta)$ we get;

$$\frac{1}{f}\frac{\partial^2 f}{\partial\xi^2}+\frac{q^2}{2}(n_j^2k^2-\beta^2)\cosh 2\xi = -\frac{1}{g}\frac{\partial^2 g}{\partial\eta^2}+\frac{q^2}{2}(n_j^2k^2-\beta^2)\cos 2\eta = h(\text{say}) \quad (5)$$

Rearranging this equation, we can write it as

$$\frac{\partial^2 f}{\partial\xi^2}-\left[h+\frac{q^2}{2}(n_1^2k^2+\beta^2)\cosh 2\xi\right]f=0 \quad (\xi<\xi_0) \quad (6a)$$

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$$\frac{\partial^2 f}{\partial \xi^2} - [h - \frac{q^2}{2}(n_2^2k^2 - \beta^2)\cosh 2\xi] f = 0 \qquad (\xi > \xi_0) \qquad (6b)$$

The general solution of equation (6) is exactly given by the modified Mathieu function

$$\psi(\xi,\eta) = I_{e\nu 1}(\xi,-\gamma_1^2)C_{e\nu 1}(\eta,-\gamma_1^2) \qquad (\xi<\xi_0, even) \qquad (7a)$$

$$\psi(\xi,\eta) = I_{o\nu 1}(\xi,-\gamma_1^2)S_{e\nu 1}(\eta,-\gamma_1^2) \qquad (\xi<\xi_0, odd) \qquad (7b)$$

$$\psi(\xi,\eta) = K_{e\nu 2}(\xi,-\gamma_2^2)C_{e\nu 2}(\eta,-\gamma_2^2) \qquad (\xi>\xi_0, even) \qquad (7c)$$

$$\psi(\xi,\eta) = K_{o\nu}(\xi,-\gamma_2^2)S_{e\nu}(\eta,-\gamma_2^2) \qquad (\xi>\xi_0, odd) \qquad (7d)$$

Due to the presence of asymmetry in an elliptical waveguide, there are two orientations of field possible, denoted by Even or Odd, depending on what combination of the Mathieu function is used. For an even wave, the axial electric and magnetic field is given by even and Odd Mathieu functions respectively. Similarly, for odd waves, the axial electric and magnetic field is given by odd and even Mathieu functions respectively.

Since here both the components E_z and are present to satisfy the boundary condition so, the modes are hybrid. Thus, we represent even hybrid modes as $e\,HE_{\nu m}$, $e\,EH_{\nu m}$ and odd modes $o\,EH_{\nu m}$, $o\,HE_{\nu m}$.

Moreover, the problem is not as simple as in the case of circular waveguides, since here the function "*I* and *K*" are not only functions of η but also the function of refractive indices n_1 and n_2.

First of all, let us take the axial component of the $e\,HE_{\nu m}$ wave:

$$H_{z1} = AI_{e\nu}(\xi,-\gamma_1^2)\cos\nu\eta \qquad (\xi \le \xi_0) \qquad (8a)$$

$$H_{z2} = LK_{e\nu}(\xi,-\gamma_2^2)\cos\nu\eta \qquad (\xi \ge \xi_0) \qquad (8b)$$

$$E_{z1} = BI_{o\nu}(\xi,-\gamma_1^2)\sin\nu\eta \qquad (\xi \le \xi_0) \qquad (8c)$$

$$E_{z2} = PK_{o\nu}(\xi,-\gamma_2^2)\sin\nu\eta \qquad (\xi \ge \xi_0) \qquad (8d)$$

After writing Maxwell's equation in elliptical coordinates, we can express the field components E_η and H_η as:

$$E_{\eta 1} = -\frac{i}{q(k^2n^2+\beta_1^2)(\cosh^2\xi-\cos^2\eta)}\left(\beta\frac{\partial E_{z1}}{\partial\eta} - \omega\mu_0\frac{\partial H_{z1}}{\partial\xi}\right) \qquad (\xi \le \xi_0) \qquad (9a)$$

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$$H_{\eta 1} = -\frac{i}{q(k^2n_1^2 + \beta^2)(\cosh^2\xi - \cos^2\eta)}\left(\beta\frac{\partial H_{z1}}{\partial\eta} - \omega n_1^2\varepsilon_0\frac{\partial E_{z1}}{\partial\xi}\right) \quad (\xi \le \xi_0) \tag{9b}$$

$$E_{\eta 2} = \frac{i}{q(k^2n_2^2 - \beta^2)(\cosh^2\xi - \cos^2\eta)}\left(\beta\frac{\partial E_{z2}}{\partial\eta} - \omega\mu_0\frac{\partial H_{z2}}{\partial\xi}\right) \quad (\xi \ge \xi_0) \tag{9c}$$

$$H_{\eta 2} = \frac{i}{q(k^2n_2^2 - \beta^2)(\cosh^2\xi - \cos^2\eta)}\left(\beta\frac{\partial H_{z2}}{\partial\eta} + \omega n_2^2\varepsilon_0\frac{\partial E_{z2}}{\partial\xi}\right) \quad (\xi \ge \xi_0) \tag{9d}$$

Putting values from equation (30) to equation (31) and solving we get

$$E_{\eta 1} = -\frac{i\cos\nu\eta}{u^2q(\cosh^2\xi - \cos^2\eta)}\{\beta B\nu I_{o\nu}(\xi,-\gamma_1^2) - \omega\mu_0 AI'_{e\nu}(\xi,-\gamma_1^2)\} \tag{10a}$$

$$H_{\eta 1} = -\frac{i\sin\nu\eta}{u^2q(\cosh^2\xi - \cos^2\eta)}\{-\beta A\nu I_{e\nu}(\xi,-\gamma_1^2) - \omega n_1^2\varepsilon_0 BI'_{o\nu}(\xi,-\gamma_1^2)\} \tag{10b}$$

$$H_{\eta 2} = -\frac{i\sin\nu\eta}{w^2q(\cosh^2\xi - \cos^2\eta)}\{-\beta L\nu K_{e\nu}(\xi,-\gamma_2^2) + \omega n_2^2\varepsilon_0 PK'_{o\nu}(\xi,-\gamma_2^2)\} \tag{10c}$$

$$E_{\eta 2} = -\frac{i\cos\nu\eta}{w^2q(\cosh^2\xi - \cos^2\eta)}\{\beta P\nu K_{o\nu}(\xi,-\gamma_2^2) - \omega\mu_0 LK'_{e\nu}(\xi,-\gamma_2^2)\} \tag{10d}$$

where $k^2n_1^2 + \beta^2 = u^2$ and $w^2 = \beta^2 - k^2n_2^2$

2.2 Modal Equation

Modal equation can be obtained by matching the boundary condition at core-cladding interface for the field components E_η, H_η, E_z and H_z, which may be written as

$E_{\eta 1} = E_{\eta 2}$ (11a)

$E_{z1} = E_{z2}$ (11b)

$H_{\eta 1} = H_{\eta 2}$ (11c)

$H_{z1} = H_{z2}$ (11d)

We will get modal equation as:

$$BI_{o\nu}(\xi,-\gamma_1^2) - PK_{o\nu}(\xi,-\gamma_2^2) = 0 \tag{12a}$$

$$AI_{e\nu}(\xi,-\gamma_1^2) - LK_{e\nu}(\xi,-\gamma_2^2) = 0 \tag{12b}$$

$$\frac{w^2}{u^2}[\beta B\nu I_{o\nu}(\xi,-\gamma_1^2) - \omega\mu_0 AI'_{e\nu}(\xi,-\gamma_1^2)] - \beta P\nu K_{o\nu}(\xi,-\gamma_2^2) + \omega\mu_0 LK'_{e\nu}(\xi,-\gamma_2^2) = 0 \tag{12c}$$

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$$\frac{w^2}{u^2}[-\beta A\nu I_{e\nu}(\xi,-\gamma_1^2)-\omega n_1^2\varepsilon_0 B I'_{o\nu}(\xi,-\gamma_1^2)]+\beta L\nu K_{e\nu}(\xi,-\gamma_2^2)-\omega n_2^2\varepsilon_0 P K'_{o\nu}(\xi,-\gamma_2^2)=0 \quad (12d)$$

Solution to equation (12a), (12b), (12c), and (12d) exists if determinant formed by these coefficients is zero.

$$\begin{vmatrix} 0 & 0 & -K_{o\nu}(\xi,-\gamma_2^2) & I_{o\nu}(\xi,-\gamma_1^2) \\ I_{e\nu}(\xi,-\gamma_1^2) & -K_{e\nu}(\xi,-\gamma_2^2) & 0 & 0 \\ -\frac{\omega\mu_0 w^2 I'_{e\nu}(\xi,-\gamma_1^2)}{u^2} & \omega\mu_0 K'_{e\nu}(\xi,-\gamma_2^2) & -\beta\nu K_{o\nu}(\xi,-\gamma_2^2) & \frac{\beta\nu w^2 I_{o\nu}(\xi,-\gamma_1^2)}{u^2} \\ -\frac{\beta\nu w^2 I_{e\nu}(\xi,-\gamma_1^2)}{u^2} & \beta\nu K_{e\nu}(\xi,-\gamma_2^2) & -\omega n_2^2\varepsilon_0 K'_{o\nu}(\xi,-\gamma_2^2) & -\frac{\omega n_1^2\varepsilon_0 w^2}{u^2} I'_{o\nu}(\xi,-\gamma_1^2) \end{vmatrix} = 0$$

Solving we get

$$\left\{\frac{1}{u^2}\frac{I'_{e\nu}(\xi,-\gamma_1^2)}{I_{e\nu}(\xi,-\gamma_1^2)}+\frac{1}{w^2}\frac{K'_{e\nu}(\xi,-\gamma_2^2)}{K_{e\nu}(\xi,-\gamma_2^2)}\right\}\left\{\frac{n_1^2}{u^2}\frac{I'_{o\nu}(\xi,-\gamma_1^2)}{I_{o\nu}(\xi,-\gamma_1^2)}+\frac{n_2^2}{w^2}\frac{K'_{o\nu}(\xi,-\gamma_2^2)}{K_{o\nu}(\xi,-\gamma_2^2)}\right\}=\left(\frac{\beta\nu}{k}\right)^2\left(\frac{V}{uw}\right)^4 \quad (13a)$$

Where $V^2 = w^2 - u^2$

Similarly, for $_oHE_{\nu m}$ modes we can find the corresponding Eigenvalue equation

$$\left\{\frac{1}{u^2}\frac{I'_{o\nu}(\xi,-\gamma_1^2)}{I_{o\nu}(\xi,-\gamma_1^2)}+\frac{1}{w^2}\frac{K'_{o\nu}(\xi,-\gamma_2^2)}{K_{o\nu}(\xi,-\gamma_2^2)}\right\}\left\{\frac{n_1^2}{u^2}\frac{I'_{e\nu}(\xi,-\gamma_1^2)}{I_{e\nu}(\xi,-\gamma_1^2)}+\frac{n_2^2}{w^2}\frac{K'_{e\nu}(\xi,-\gamma_2^2)}{K_{e\nu}(\xi,-\gamma_2^2)}\right\}=\left(\frac{\beta\nu}{k}\right)^2\left(\frac{V}{uw}\right)^4 \quad (13b)$$

Numerical solution of two equations can be found by using expansion of Mathieu function in terms of a series of Bessel function. For small eccentricity the series may be terminated after the term of order of e^2 which makes the computation comparatively easier.

Mode cutoffs are still found in the limit of $\beta = kn_2$

The only mode with zero cutoff are $_oHE_{11}$ and $_eHE_{11}$, which are therefore the fundamental mode of elliptical fiber.

By analogy with circular fiber higher order mode is expected to be associated with ν = 0 in (13a)

and (13b). Since the function $I(\xi,\gamma^2)$ does not exist from (13a) and (13b) we see that there is only $_oTE_0$ and $_eTM_0$ exists, having characteristic equation:

$$F(\beta) = -\frac{1}{u^2}\frac{I'_{o0}(\xi,-\gamma_1^2)}{I_{o0}(\xi,-\gamma_1^2)}+\frac{1}{w^2}\frac{K'_{o0}(\xi,-\gamma_2^2)}{K_{o0}(\xi,-\gamma_2^2)}=0 \quad \text{for } {}_oTE_0 \quad (14a)$$

$$F(\beta) = \frac{n_1^2}{u^2}\frac{I'_{e0}(\xi,-\gamma_1^2)}{I_{e0}(\xi,-\gamma_1^2)}+\frac{n_2^2}{w^2}\frac{K'_{e0}(\xi,-\gamma_2^2)}{K_{e0}(\xi,-\gamma_2^2)}=0 \quad \text{for } {}_oTM_0 \quad (14b)$$

$K'_{o0}(\xi,-\gamma_2^2)$

Since $K_{o0}(\xi,-\gamma_2^2)$ is always negative equation (14a) cannot be satisfied.

Now cutoff condition can be found by putting $w \to 0$

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$$\frac{I'_{e0}(\xi,-\gamma_1^2)}{I_{e0}(\xi,-\gamma_1^2)} = 0$$

Integrating both side and simplifying we get

$$I_{e0}(\xi,-\gamma_1^2) = 0 \qquad (14c)$$

This is the cutoff condition for $_oTM_0$, so the waveguide supports odd *TM* mode. The equations (14a), (14b), (14c) hold good for weakly guided elliptical waveguides with small eccentricity, and the core is filled with the material of negative refractive index.

3. NUMERICAL COMPUTATION, RESULTS, AND DISCUSSION

The Eigenvalue equation (13b) known as characteristic equation is the most important equation of our analysis because it gives all information regarding the dispersion relation of the proposed waveguide. We note at this stage that the values of *u*, *w*, Ko_0 are always positive therefore in equation (14a) the odd TE mode solutions cannot exist because are negative. For odd TM mode solutions, we take a particular value of L.H.S. and compute the left-hand side (LHS) of equation (14b) for many equi-spaced values of β/k lying between ∞ to n_2. When $\beta/k \rightarrow n_2$ then $y_2^2 \rightarrow 0$ and L.H.S of equation (14b) becomes $F(\beta) \rightarrow -\infty$. Similarly, when $\beta/k \rightarrow \infty$, both $y_1^2 \rightarrow \infty$, $y_2^2 \rightarrow \infty$ and the L.H.S. of equation (14b) becomes $F(\beta) > 0$. Here the change of sign of the continuous function $F(\beta)$ over the range, n2, ∞ shows that at least one solution must exist. This finding establishes the theoretical foundation for the existence of wave modes in the waveguide. However, while our numerical computations consistently yield a single solution, we do not have formal proof of its uniqueness. Further analysis may be required to determine whether multiple solutions are possible under different conditions. Nonetheless, this result validates the feasibility of wave propagation in the proposed elliptical waveguide and provides a basis for further numerical and experimental studies. Hence, all our numerical calculations have given a single solution. However, we have no formal proof of the uniqueness of the solution. It may be interesting to discuss the behaviour of this solution in extreme cases. When, we have $y_1^2, y_2^2 \rightarrow \infty$ and we can again use the asymptotic equivalence, so that the dispersion equation (14b) becomes $\frac{n_1^2}{u^2} = \frac{-n_2^2}{w^2}$. But the dispersion

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$V \to 0$ $\beta/k \to \infty$ $y^2 = y^2 = q\beta$ $Ie'(\xi,-y^2)$ $Ke'(\xi,-y^2)$

1 2 equation for $n^2 \frac{}{0\ 1} = n^2 \frac{}{0 \qquad 2}$,with and becomes $1\ Ie\ {}_0(\xi,-y^2)_1\ \ 2\ Ke\ {}_0(\xi,-y^2)_2$,

$\frac{\beta}{} = \frac{1\ Ke'(\xi,-y^2)}{0 \qquad 2}$ $\sqrt{n_1^2 + n_2^2}$. Our numerical calculations consistently

which explicitly gives $k\ \ V\ Ke(\xi,-y^2)\ {}_0 \qquad {}_2$

yield a single solution for the dispersion equation; however, we do not have a formal mathematical proof guaranteeing the uniqueness of this solution. While the numerical results suggest that only one solution exists, further theoretical analysis may be required to determine whether multiple solutions could arise under different conditions. To gain deeper insights, it is useful to examine the behaviour of this solution in extreme cases, where certain parameters approach their limiting values.

By considering these extreme cases, we can utilize asymptotic approximations to simplify the dispersion equation. This approach allows us to understand how the wave propagation characteristics change when parameters reach their extreme limits. Applying asymptotic equivalence, we transform the dispersion equation (14b) into a simpler form, making it easier to analyze. Furthermore, when we consider a different scenario with specific constraints on the parameters, the dispersion equation simplifies even further, leading to an explicit expression for the solution.

This explicit solution provides valuable insights into the fundamental behavior of wave modes within the waveguide. It confirms that under certain limiting conditions, the wave propagation characteristics become predictable and can be described using simplified mathematical expressions. Despite these findings, the absence of formal proof for uniqueness remains an open question, and further theoretical studies may be required to rigorously establish whether the solution obtained is truly unique or if additional solutions might exist under varying conditions. Now we are in a position to make some numerical calculations by using the cutoff equation (14c)

and by choosing $n_1 = 1.5$, $n_2 = 1.3$, $\lambda_0 = 1.55\mu m$. We define the V-parameter as $V = \frac{2\pi q}{\lambda_0}\sqrt{n_1^2 + n_2^2}$.

Here we want to point out that this choice of the V-parameter is not unique but is the simplest. We obtain the cutoff V-values for a mode that can be seen directly from the dispersion curve or obtained from the cutoff equation (14c) by allowing $W \to 0$. Fig.3 (a-e) show the cutoff V-values for odd TM mode for negative refractive index material loaded elliptical core fiber with aspect ratios 0.111, 0.333, 0.5, and 0.666, 0.996 respectively.

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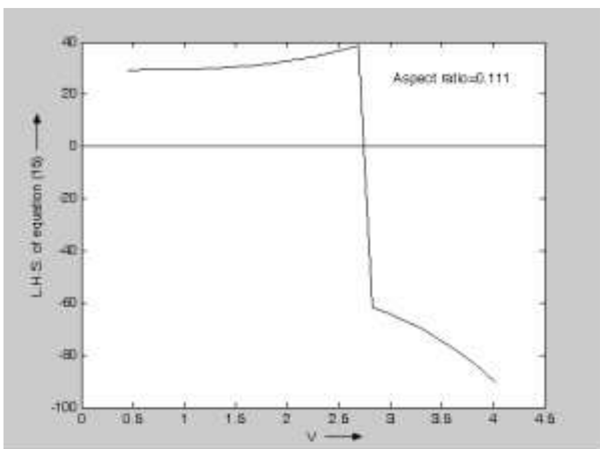
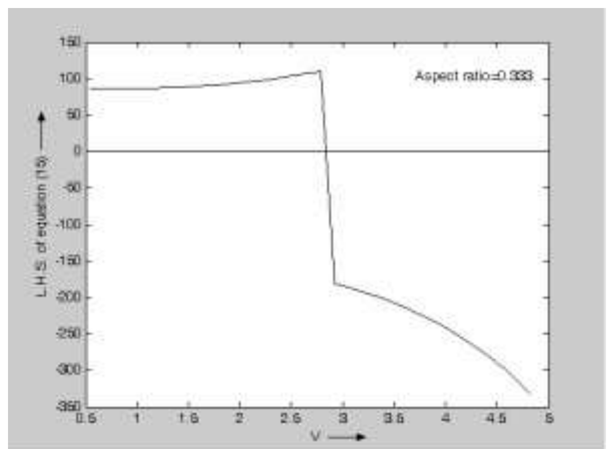
Now that we have established the theoretical framework, we proceed with numerical calculations using the cutoff equation (14c). For this analysis, we select specific parameter values: refractive indices $n_1 = 1.5$, $n_2 = 1.3$, along with a free-space wavelength $\lambda_0 = 1.55\mu m$. To simplify the computations, we define the V-parameter, $V = \frac{2\pi q}{\lambda_0}\sqrt{n_1^2 + n_2^2}$ a key dimensionless quantity that characterizes the waveguide's behaviour. It is important to note that while this definition of the V-parameter is not unique, it is chosen because it provides the simplest formulation for our calculations. To determine the cutoff conditions for the waveguide modes, we compute the cutoff V-values. These values indicate the threshold at which a mode can propagate. The cutoff V- values can be obtained in two ways: by directly extracting them from the dispersion curve or by solving the cutoff equation (14c) under the condition $W \to 0$. The results are illustrated in Figures 3a to 3e, which show the cutoff V-values for the odd TM mode in an elliptical core fiber loaded with a negative refractive index material. The calculations are performed for different aspect ratios, specifically 0.111, 0.333, 0.5, 0.666, and 0.996. These aspect ratios allow us to analyze how changes in the core geometry influence the wave propagation and cutoff conditions, providing valuable insights into the optical behavior of the waveguide.



(a)



(b)

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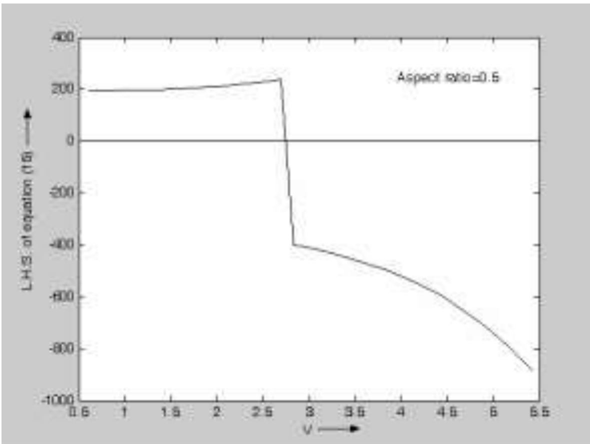
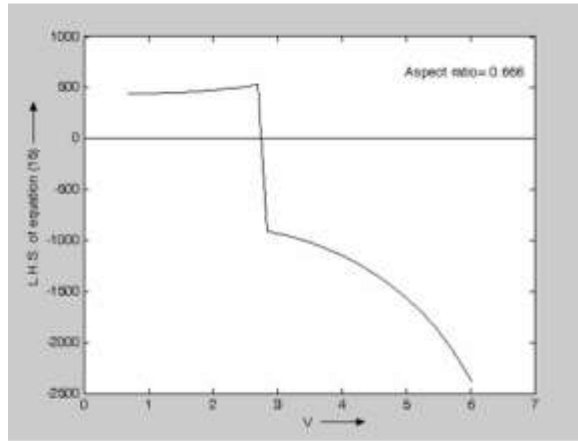
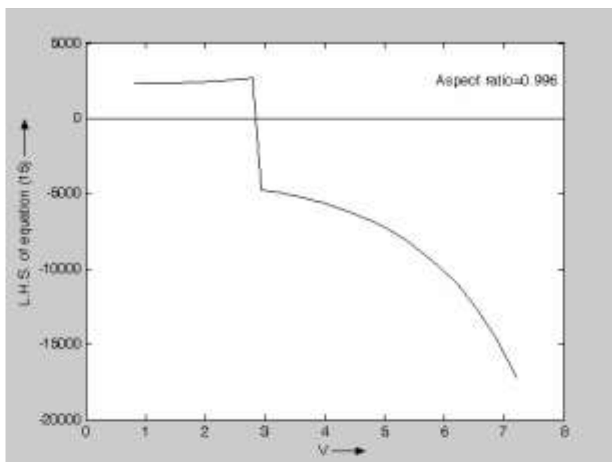


(c)



(d)



(e)

Figure 3: Cutoff V-values for the odd TM mode in a negative refractive index material-loaded elliptical core fiber for different aspect ratios: (a) 0.111, (b) 0.333, (c) 0.5, (d) 0.666, and (e) 0.996.

A key observation from our numerical results is that the introduction of NRIM significantly alters the dispersion characteristics of the waveguide. Unlike conventional waveguides, where higher order modes appear at well-defined V-thresholds, the presence of NRIM modifies these thresholds due to the effective negative permittivity and permeability contributions. Additionally, as the aspect ratio increases, the cutoff V-values exhibit a non-linear variation, suggesting a complex interplay between wave confinement and structural anisotropy.

These findings have significant implications for designing waveguide-based optical devices, particularly in sensor applications where modal tunability is crucial. The ability to engineer cutoff conditions via NRIM loading opens new possibilities for optimizing waveguide performance in fields such as biosensing, telecommunications, and integrated photonics. Future studies may further explore the impact of additional parameters, including higher-order modal interactions and temperature-dependent refractive index variations, to refine the waveguide design for practical applications. These cutoff V-values and their dependence on the aspect ratio for odd TM mode can be seen for the proposed waveguide in Table 1. The variation of the cutoff value with the aspect ratio demonstrates a decreasing trend when the aspect ratio is increased from 0.3 to 0.5. This indicates that as the waveguide core elongates, the mode confinement changes, resulting in a shift in the cutoff condition. Furthermore, we observe that the variation in cutoff values for the aspect ratio remains relatively small, suggesting a stable mode structure under slight structural changes.

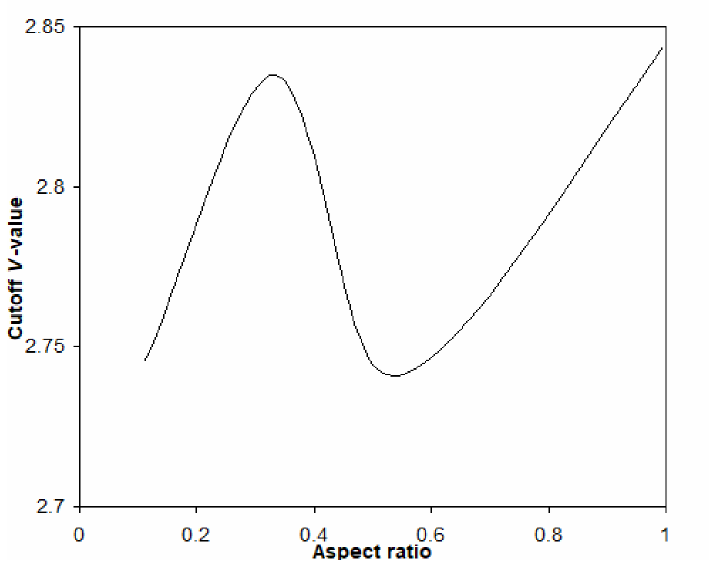


Fig. 4: Variation of cutoff V –values with aspect ratio for lower order mode

For a more comprehensive analysis, Table 1 is supplemented by corresponding graphical representations shown in Fig.4. These figures provide a visual insight into how the cutoff V- values shift with aspect ratio variations, highlighting the underlying trend. Additionally, we present the variation of the effective index with V-values. This representation is particularly useful for understanding how the guided modes behave under different excitation conditions. By analyzing the effective index variations, we can infer the influence of NRIM loading on wave propagation, which is essential for designing tunable photonic devices. The graphical interpretations, along with numerical values, reinforce the conclusions drawn from the theoretical formulation and numerical computations, confirming the consistency of the results.

4. CONCLUSION

In this paper, a novel elliptical core fiber loaded with negative refractive index material (NRIM) has been proposed and analyzed using the method of separation of variables in the elliptical coordinate system. The eigenvalue equation, formulated in terms of first and second-kind Mathieu functions, is derived under the weak guidance condition. The cutoff frequency for lower-order modes at various aspect ratios has been computed and analyzed, revealing several significant findings. Our analysis confirms that for, the waveguide supports only odd TM modes, indicating a strong mode-selective behavior. Additionally, as the aspect ratio increases from 0.3 to 0.5, the cutoff frequency exhibits a decreasing trend. This relationship suggests that aspect ratio tuning can serve as an effective design parameter for controlling mode propagation within the fiber. The variation of modal cutoff V-values for odd TM modes concerning the aspect ratio is summarized in Table 1, supplemented by graphical representations in Figures 3a to 3e. These results emphasize that increasing the aspect ratio leads to a slight yet systematic variation in cutoff conditions. Notably, the results indicate that for applications requiring fewer supported modes, selecting an optimal aspect ratio can be beneficial, with specific configurations offering the most favourable conditions for single-mode operation.

The insights gained from this study provide a foundation for further exploration of NRIM-loaded waveguides, with potential applications in advanced optical communication systems, sensing technologies, and photonic integrated circuits. Future work could explore higher-order modal interactions, temperature-dependent effects, and further optimization of the fiber structure for tailored photonic applications.

Table 1. Variation of modal cutoff V-values with aspect ratio for odd TM mode

S. No.	Aspect ratio	cutoff frequency
1.	0.111	2.7456
2.	0.333	2.8335
3.	0.5	2.7444
4.	0.666	2.7586
5.	0.996	2.8439

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